On Measuring Connection Strengths and Link Strengths in Discrete Bayesian Networks

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Abstract

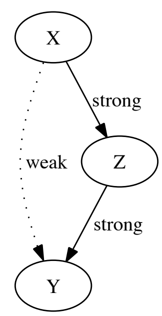
This paper discusses measures for connection strength (strength between any two nodes) and link strength (strength along a specific edge) in Discrete Bayesian Networks. Existing measures for connection strength are first reviewed. Then a little-known link strength measure – originally proposed for the sole purpose of speeding up approximate inference (Jitnah and Nicholson 1998) – is turned into two different link strength measures suitable for the *interpretation* of Bayesian Networks. Additional new measures, Mutual Information Percentage and Link Strength Percentage, are then proposed. Finally several properties of the measures are discussed and highlighted through numerical examples.

1 INTRODUCTION

The graph of a Bayesian Network provides a very intuitive tool to communicate the (causal) relationships between the network's variables to a user. However, for further interpretation of the network it is helpful to visualize not only the *existence* of arrows, but also the *strength* of the various connections.

Boerlage was the first to formally introduce the concepts of link strength versus connection strength for Bayesian Networks with binary nodes (Boerlage 1992). Boerlage defines *connection strength* for any pair of nodes (adjacent or not) to measure the strength between those nodes taking any possible path between them into account. In contrast *link strength* (also known as *arc weight*) is defined for a specific edge and measures the strength of connection only along that single edge.

To demonstrate the difference between these concepts in particular for adjacent nodes consider the network



X :
$P(X = True) = 0.5$
Z :
$P(Z = True|X = True) = 0.9$
$P(Z = True|X = False) = 0.1$
Y :
$P(Y = True|X = True, Z = True) = 0.9$
$P(Y = True|X = False, Z = True) = 0.89$
$P(Y = True|X = True, Z = False) = 0.1$
$P(Y = True|X = False, Z = False) = 0.11$

Figure 1: Sample BN with weak link from X to Y, but strong links from X to Z and Z to Y.

in Figure 1. Each of the three nodes only has two states, $True$ and $False$. Let us focus on the connection between nodes X and Y. For this sample network the *direct* link from X to Y is weak[1] , while the indirect link from X to Y *through* Z is very strong. According to the above (vague) concept definitions, the connection strength, CS, between X and Y is strong here, but the link strength, LS, of the edge $X \rightarrow Y$ is weak:

$$\begin{aligned} \text{CS}(X, Y) &= \text{strong}, \\ \text{LS}(X \rightarrow Y) &= \text{weak}. \end{aligned}$$

Any pair of measures for link strength and connection strength should yield this result for the above example.

1.1 OVERVIEW OF MEASURES

The following measures are discussed in this paper:
(1) Entropy (Shannon 1949) is used to measure the uncertainty in a single node.
(2) Mutual information (Shannon 1949, Pearl 1988) is used to measure connection strength.
(3) Two variations of the link strength measure in

[1] This becomes obvious by noting that the state of X has little effect on the values of $P(Y = True|X, Z)$.

(Nicholson and Jitnah 1998) are proposed: *True Average Link Strength* and *Blind Average Link Strength*. (4) Mutual Information *Percentage* and Link Strength *Percentage* are proposed to measure the *percentage* of the existing uncertainty that has been removed.
All measures are defined in this document only for *discrete* Bayesian Networks.

1.2 LITERATURE

The most important work related to this article, as evident from the previous sections, consists of Shannon's definitions of Entropy and Mutual Information (Shannon 1949), Pearl's use of Mutual Information for Bayesian Networks (Pearl 1988), Boerlage's definition of link strength and connection strength for Bayesian networks with binary nodes (Boerlage 1992) and Jitnah and Nicholson's definition of a specific link strength measure for fast approximate inference (Jitnah and Nicholson 1998).

Other work of interest – although not used here – is the work by Lacave and Diez (Lacave and Diez 2004) proposing a measure for the "magnitude of influence" of two ordinal variables and displaying it by the thickness of an arc. Other visualization techniques are reviewed in (Lacave and Diez 2002) and (Zapata-Riviera et al. 1999), but no other measures for link or connection strengths are presented in those articles.

2 ENTROPY AS UNCERTAINTY MEASURE

Entropy was already defined by Shannon in the late 1940s (Shannon 1949) and has become the most commonly used measure for the uncertainty of a random variable. It is also the basis for all connection and link strength measures discussed in this paper and thus deserves special consideration.

Definition The *entropy* of a discrete random variable, X, is defined as

$$U(X) = \sum_{x_i} P(x_i) \log_2 \frac{1}{P(x_i)}. \qquad (1)$$

Some readers may be more familiar with the expression $U(X) = -\sum_{x_i} P(x_i) \log_2 P(x_i)$, which is identical to (1).

Interpretation How much uncertainty is there in X if no evidence is given for any of the nodes?

2.1 WEAKNESSES OF ENTROPY

Although no better alternative has yet emerged to measuring uncertainty, entropy is also known to have some weaknesses. Pearl explained a weak point of using entropy – or any other measure that is a function of only the *probabilities* of a random variable's states – to measure uncertainty (Pearl 1988, pp. 322-323):

The main weakness of Shannon's measure is that it does not reflect the ordering or scale information relative to the values that a variable may take. For example, the uncertainty associated with the belief "The temperature is between 37^o *and* 39^o *would have the same entropy measure as the uncertainty associated with "The temperature is either between* 0^o *and* 1^o *or between* 99^o *and* 100^o*" (assuming uniform distribution over the intervals specified). Entropy is invariant to reordering or renaming the values in the domain, so it cannot reflect the fact that we perceive an error between* 37^o *and* 38^o *to be much less critical than an error between* 0^o *and* 100^o*. [...]*

The source of this peculiar behavior is that entropy, contrary to folklore, does not measure the harm caused by uncertainty; it measures the cost of removing the uncertainty (by querying an oracle and paying the same fee for all binary queries). This is why Shannon's mutual information measure endows equal penalty to all errors.

Uffink (Uffink 1995) provides additional criticism of entropy as unique information measure.

2.2 ADDITIONAL INTERPRETATION

There are many other interpretations of entropy. One can also think of the term $\left(\log \frac{1}{P(x_i)}\right)$ as describing the *surprise* if event x_i occurs. Then $U(X)$ is the *expected (average) surprise*, if infinitely many trials are performed. However, *that* interpretation does not explain the logarithmic scale, since one can think of many measures of *surprise* that are not logarithmic.

In contrast, Pearl's interpretation above of entropy *as the approximate number of required binary queries to determine the state of the variable* seems to be the *only* one that fully explains the logarithmic scale and thus the exact formula of entropy. (Pearl does not provide a derivation of this interpretation in (Pearl 1988), but it is actually fairly easy to derive.)

3 MEASURES FOR CONNECTION STRENGTH

Connection strength between X and Y measures how strongly information on the state of X affects the state of Y (and vice versa). The standard approach is to compare the distribution of Y *without* any evidence to the distribution of Y if there *is* evidence for X. Mutual Information is the most common implementation of this idea: one simply calculates $U(Y)$ and $U(Y|X)$ and compares them (see Section 3.1).

An alternative is to apply a divergence measure between the two probability distributions of Y and $Y|X$. For example, in their earlier work Nicholson and Jitnah apply the Bhattacharyya distance (Nicholson and Jitnah 1997) to the distributions. However, that approach yields less suitable results than Mutual Information (Nicholson and Jitnah 1998).

3.1 MUTUAL INFORMATION

Shannon (Shannon 1949) introduced Mutual Information for the purpose of communication theory. Pearl (Pearl 1988) was the first to propose the use of mutual information to measure connection strength in Bayesian Networks to determine the relevance of some nodes on others.

Definition *Mutual Information* is defined as

$$MI(X,Y) = U(Y) - U(Y|X), \tag{2}$$

where $U(Y|X)$ is calculated by averaging $U(Y|x_i)$ over all possible states x_i of X, taking $P(x_i)$ into account:

$$U(Y|X) = \sum_{x_i} P(x_i) U(Y|x_i). \tag{3}$$

Simple arithmetic transformations yield the formula:

$$MI(X,Y) = \sum_{x,y} P(x,y) \log_2 \left(\frac{P(x,y)}{P(x)P(y)} \right).$$

Mutual Information is symmetric in X and Y, i.e. $MI(X,Y) = MI(Y,X)$.

Interpretation How much is the uncertainty in Y reduced by knowing the state of X? How much is the uncertainty in X reduced by knowing the state of Y?

3.2 MUTUAL INFORMATION PERCENTAGE

In some cases the *absolute amount* of uncertainty reduction in a variable may provide less insight than the *percentage* of the original uncertainty that was removed. Thus we propose a new measure, Mutual Information Percentage, to be used in conjunction with Mutual Information.

Definition *Mutual Information Percentage* is defined for $U(Y) \neq 0$ as

$$MI\%(X,Y) = \frac{MI(X,Y)}{U(Y)} \cdot 100 = \frac{U(Y) - U(Y|X)}{U(Y)} \cdot 100.$$

Mutual Information is *not* symmetric in X and Y, i.e. $MI\%(X,Y) \neq MI\%(Y,X)$. $MI\%(X,Y)$ is undefined for $U(Y) = 0$, which makes perfect sense: if there is zero uncertainty to begin with, then it makes no sense to ask what percentage of it was removed.

Interpretation By how many percentage points is uncertainty in Y reduced by knowing the state of X?

4 MEASURES FOR LINK STRENGTH

There is much less literature on link strength than on connection strength and it appears to be harder to measure. Boerlage (Boerlage 1992) defined measures for both link strength and connection strength. However, those only apply to two-state variables and are not used here. (Nicholson and Jitnah 1998) and (Jitnah 1999) derived expressions for link strength based on mutual information for the purpose of efficient approximate inference. Variations of those expressions are used here.

4.1 TRUE AVERAGE LINK STRENGTH

A definition of link strength of an edge $X \to Y$ can be derived from the definition of connection strength. When considering a link $X \to Y$, we need to decide how to deal with the *other* parents of Y in order to focus on the connection from parent X to child Y *solely* along edge $X \to Y$. The approach used here is to instantiate all *other* parents of Y, leaving the direct connection from X to Y as only pathway through which information can travel from X to Y.

Denoting the set of *other* parents of Y as $\mathbf{Z} = \{Z_1, \ldots Z_n\}$, we can adjust Equation (2) of Mutual Information by conditioning both terms on the right on $\mathbf{Z}$, resulting in the following definition. (We use boldface for $\mathbf{Z}$ and $\mathbf{z}$ to indicate that each represents a *set* of zero, one or more variables.)

Definition *True Average Link Strength* is defined as

$$LS^{true}(X \to Y) = U(Y|\mathbf{Z}) - U(Y|X,\mathbf{Z}),$$

where $U(Y|X,\mathbf{Z})$ is the average over the states of all parents and is defined as

$$U(Y|X,\mathbf{Z}) = \sum_{x,\mathbf{z}} P(x,\mathbf{z}) U(Y|x,\mathbf{z}) \tag{4}$$
$$= \sum_{x,\mathbf{z}} P(x,\mathbf{z}) \sum_{y} P(y|x,\mathbf{z}) \log_2 \left(\frac{1}{P(y|x,\mathbf{z})} \right),$$

and $U(Y|\mathbf{Z})$ is defined analogously as the average over all *other* parents:

$$U(Y|\mathbf{Z}) = \sum_{\mathbf{z}} P(\mathbf{z}) U(Y|\mathbf{z}), \tag{5}$$

where $\mathbf{z}$ represents all possible state *combinations* of the set of other parents, $\mathbf{Z}$.

Using (4) and (5) and some transformations yields

$$LS^{true}(X \rightarrow Y) = \sum_{x,\mathbf{z}} P(x,\mathbf{z}) \sum_{y} P(y|x,\mathbf{z}) \log_2 \frac{P(y|x,\mathbf{z})}{P(y|\mathbf{z})}. \quad (6)$$

Interpretation By how much is the uncertainty in Y reduced by knowing the state of X, if the states of all other parent variables are known (averaged over the parent states using their *actual* joint probability)?

4.2 COMPARISON TO LINK STRENGTH BY NICHOLSON AND JITNAH

Nicholson and Jitnah proposed the following formula for link strength (Nicholson and Jitnah 1998):

$$\omega(X,Y) = \sum_{k \in \Omega(\mathbf{Z})} p_{pr}(\mathbf{Z}=\mathbf{k}) \sum_{i \in \Omega(X)} p_{pr}(X=i) \sum_{j \in \Omega(Y)} p(Y=j|X=i,\mathbf{Z}=\mathbf{k}) \log \frac{p(Y=j|X=i,\mathbf{Z}=\mathbf{k})}{p_{pr}(Y=j|\mathbf{Z}=\mathbf{k})},$$

where $\Omega(X)$ represents all discrete states of X, p is the probability function and p_{pr} is an approximation of probability that avoids using any inference in the network. Avoiding inference is crucial in their work since they use the measure to identify the most relevant arcs and nodes for a specific inference query to speed up that query.

Ignoring the approximations (i.e. the subscripts of p_{pr}) and converting to our notation, Nicholson and Jitnah's formula can be written as

$$\omega(X,Y) = \sum_{x,\mathbf{z}} P(\mathbf{z})P(x) \sum_{y} P(y|x,\mathbf{z}) \log_2 \frac{P(y|x,\mathbf{z})}{P(y|\mathbf{z})},$$

which is identical to Eq. (6) except that Jitnah and Nicholson assume $P(x,\mathbf{z}) = P(x)P(\mathbf{z})$ in their formula – an approximation required to avoid using inference in their calculation. In comparison, Equation (6) of True Average Link Strength yields exact calculation without regard to computational complexity. Section 6.6 provides a few numerical results for both formulas.

4.3 TRUE AVERAGE LINK STRENGTH PERCENTAGE

Just as percentage of uncertainty reduction can be important in mutual information, the same holds for True Average Link Strength. Therefore we propose the new measure of True Average Link Strength Percentage to be used in conjunction with True Average Link Strength.

Definition *True Average Link Strength Percentage* is defined for $U(Y|\mathbf{Z}) \neq 0$ as

$$LS\%^{true}(X \rightarrow Y) = \frac{LS^{true}(X \rightarrow Y)}{U(Y|\mathbf{Z})} \cdot 100 = \frac{U(Y|\mathbf{Z}) - U(Y|X,\mathbf{Z})}{U(Y|\mathbf{Z})} \cdot 100. \quad (7)$$

Analogously to $MI\%$, $LS\%^{true}(X \rightarrow Y)$ is undefined if $U(Y|\mathbf{Z}) = 0$ (for the same reason as $MI\%$).

Interpretation By how many percentage points is the uncertainty in Y reduced by knowing the state of X, if the states of all other parent variables are known (averaged over the parent states using their *actual* joint probability)?

4.4 BLIND AVERAGE LINK STRENGTH

This measure is derived from True Average Link Strength by disregarding the actual frequency of occurrence of the parent states. Namely we assume that $X, \mathbf{Z}$ are independent and all uniformly distributed:

$$\hat{P}(x,\mathbf{z}) = P(x)P(\mathbf{z}),\ \hat{P}(x) = \frac{1}{\#(X)},\ \hat{P}(\mathbf{z}) = \frac{1}{\#(\mathbf{Z})}, \quad (8)$$

where $\#(X)$ denotes the number of discrete states of X, etc. This assumption creates a local measure that depends only on the child node and its conditional probability table, but nothing else in the network.

Definition *Blind Average Link Strength* is defined as

$$LS^{blind}(X \rightarrow Y) = \hat{U}(Y|\mathbf{Z}) - \hat{U}(Y|X,\mathbf{Z}),$$

where

$$\hat{U}(Y|\mathbf{Z}) = \frac{1}{\#(X)\#(\mathbf{Z})} \sum_{x,y,\mathbf{z}} P(y|x,\mathbf{z}) \log_2 \frac{\#(X)}{\sum_x P(y|x,\mathbf{z})},$$

$$\hat{U}(Y|X,\mathbf{Z}) = \frac{1}{\#(X)\#(\mathbf{Z})} \sum_{x,y,\mathbf{z}} P(y|x,\mathbf{z}) \log_2 P(y|x,\mathbf{z}).$$

Note that $\hat{U}(Y|\mathbf{Z})$ and $\hat{U}(Y|X,\mathbf{Z})$ are obtained from $U(Y|\mathbf{Z})$ and $U(Y|X,\mathbf{Z})$ simply by applying assumptions (8). This definition yields the simple formula

$$LS^{blind}(X \rightarrow Y) = \frac{1}{\#(X)\#(\mathbf{Z})} \sum_{x,y,\mathbf{z}} P(y|x,\mathbf{z}) \log_2 \left(\frac{P(y|x,\mathbf{z})}{\frac{1}{\#(X)} \sum_x P(y|x,\mathbf{z})} \right),$$

where $P(y|x,\mathbf{z})$ is given by the conditional probability table of Y and *no* inference is required at all.

Interpretation By how much is the uncertainty in Y reduced by knowing the state of X, if the states of all other parent variables are known (averaged over the parent states assuming all parents are independent of each other and uniformly distributed)?

Comment: This is the simplest and computationally least expensive measure. It is also a local measure, taking only the child and *its* conditional probabilities into account, thus allowing for isolated analysis of child and parents, regardless of the rest of the network[2].

4.5 BLIND AVERAGE LINK STRENGTH PERCENTAGE

Applying independence and uniformity assumptions (8) to the True Average Link Strength Percentage (7) yields the Blind Average Link Strength Percentage.

Definition *Blind Average Link Strength Percentage* is defined for $\hat{U}(Y|\mathbf{Z}) \neq 0$ as

$$LS\%^{blind}(X \to Y) = \frac{LS^{blind}(X \to Y)}{\hat{U}(Y|\mathbf{Z})} \cdot 100 = \frac{\hat{U}(Y|\mathbf{Z}) - \hat{U}(Y|X,\mathbf{Z})}{\hat{U}(Y|\mathbf{Z})} \cdot 100.$$

Analogously to $MI\%$, $LS\%^{blind}(X \to Y)$ is undefined if $\hat{U}(Y|\mathbf{Z}) = 0$ (for the same reason as $MI\%$).

Interpretation By how many percentage points is the uncertainty in Y reduced by knowing the state of X, if the states of all other parent variables are known (averaged over the parent states assuming all parents are independent of each other and uniformly distributed)?

5 COMPUTATIONAL ISSUES

5.1 HANDLING DEGENERATE CASES

Considering the formulas for entropy, mutual information and link strengths turns up a variety of potential degenerate cases that would lead to either (1) Division by zero (such as in $\left(p \log_2 \frac{1}{p}\right)$ if $p = 0$); (2) Calculating the logarithm of zero; or (3) Calculating an undefined expression such as $P(y|x)$ for $P(x) = 0$. Fortunately, careful analysis shows that in all of those cases the expressions in question converge towards zero when approaching the degenerate case. For example it is

$$\lim_{p \to 0} \; p \; \log_2 \frac{1}{p} \; = \; 0.$$

[2]In comparison the measure in (Nicholson and Jitnah 1998) is not local, since a change in the parents' probabilities *may* change their link strength value.

Thus in this case it is sufficient to simply check whether $p < \epsilon$ (we use $\epsilon = 10^{-10}$) and in that case to set the entire expression to zero. This avoids numerical error for $p \approx 0$ and division-by-zero error for $p = 0$. A similar procedure can be used for all degenerate cases: whenever certain probabilities are smaller than ϵ the corresponding expression is treated as zero.

5.2 COMPUTATIONAL COMPLEXITY

The computation with the highest computational complexity in all of the connection strength and link strength formulas appears to be the inference used to calculate the various required joint probabilities. (In our implementation all inference is performed using the Junction Tree method.) The inference requirements are as follows:

- $CS(X,Y)$ requires $P(X,Y)$.
- $LS^{true}(X \to Y)$ requires P(all parents of Y).
- $LS^{blind}(X \to Y)$ requires no inference at all.
- Each percentage measure requires the same probabilities as the corresponding absolute measure above.

6 MORE PROPERTIES AND INTERPRETATION

This section provides additional intuition on the measures by presenting some properties and illustrating them by several examples. All the measures discussed in the previous sections plus several visualization routines were implemented as the LinkConnectionStrength package (Ebert-Uphoff 2006) for Intel's Open-Source Probabilistic Network Library (PNL). (Sources and documentation for the LinkConnectionStrength Package are available to the public at www.DataOnStage.com.)

6.1 DO OUR MEASURES BEHAVE AS DESIRED?

Let us revisit the network from Section 1 (Figure 1) used to demonstrate the desired difference in behavior between connection strength and link strength and see whether the measures defined here actually behave in the desired way. Table 1 shows the results for the network from Figure 1 for True Average and Blind Average Link Strength for each edge, as well as Mutual Information for each node pair. The values are consistent with the expectations for link strength and connection strength specified in Section 1, specifically:

Link Strength: No matter which formula is used (True Average or Blind Average), the link strengths of the arcs from X to Z and from Z to Y are significant, while the strength of arc $X \to Y$ nearly vanishes.

Table 1: Results for Sample Network in Figure 1

	LS^{true}	LS^{blind}	MI
$X \to Y$	0.000	0.000	0.311
$X \to Z$	0.531	0.531	0.531
$Z \to Y$	0.204	0.516	0.515

Connection Strength: Each pair of nodes, (X, Y), (X, Z) and (Y, Z), is strongly connected. In particular, the pair of nodes (X, Y) receives a strong connectivity value, because X and Y are strongly connected through the chain $X \to Z \to Y$.

6.2 MUTUAL INFORMATION VERSUS TRUE AVERAGE LINK STRENGTH

This section discusses in in which cases Mutual Information and True Average Link Strength coincide.

Proposition 6.1 *For any node, Y, that only has one parent, X, mutual information and True Average Link Strength yield the same value, i.e.*

$$MI(X, Y) = LS^{true}(X \to Y) \quad \textit{if } Pa(Y) = \{X\}.$$

Mutual information Percentage and True Average Link Strength Percentage also coincide in this case,

$$MI\%(X, Y) = LS\%^{true}(X \to Y) \quad \textit{if } Pa(Y) = \{X\}.$$

Proof This follows directly by using that $\mathbf{Z}$ is the empty set in the definition of LS^{true} and $LS\%^{true}$.

Note: A similar result was demonstrated for Nicholson and Jitnah's measure in (Nicholson and Jitnah 1998).

In contrast, let us consider the node pair (X, Y) in the simple 3-node network

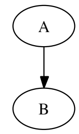
$$X \to Y \leftarrow Z.$$

Since the *only* causal connection between X and Y is the arc $X \to Y$, one may initially expect that Mutual Information and True Average Link Strength would also coincide for that arc. However, mutual information measures how much uncertainty is removed from Y by knowing the state of X *if nothing else is known.* In contrast, True Average Link Strength measures how much uncertainty is removed from Y by knowing the state of X *if the state of Z is known.*

In summary, Mutual Information and True Average Link Strength generally only coincide if the child has only one parent.

6.3 WHICH NUMBERS INDICATE A "STRONG" RELATIONSHIP?

This question cannot be fully answered here, but we try to shed some light on it by considering the trivial example in Figure 2. Nodes A and B are both binary with states $True$ and $False$ and b is a free parameter. Note that for this trivial system it is $MI(A, B) = LS^{true}(A \to B) = LS^{blind}(A \to B)$, since B only has a single, uniformly distributed parent (A).



$$\mathbf{A}: \; P(A = True) = 0.5$$
$$\mathbf{B}: \; P(B = True | A = True) = b$$
$$P(B = True | A = False) = 1 - b$$

Figure 2: Two-Node Network with parameter b.

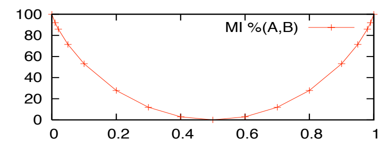


Figure 3: $MI\%(A, B)$ vs. b for Network in Figure 2.

How do mutual information and link strength "scale" for this network, i.e. what values do they result in for varying b? Table 2 (on next page) and Figure 3 show results for $MI(A, B)$ and $MI\%(A, B)$ – and thus also for $LS^{true/blind}(A \to B)$ and $LS\%^{true/blind}(A \to B)$ – for a variety of values of b. Notice *how* quickly $MI(A, B)$ decreases when increasing b from zero. For example, for $b = 0.1$ we know that in 90% of cases B is $True$ if and only if A is $False$. However, the connection/link strength value is only 0.531 with a percentage value of 53.1%. Similarly, even for $b = 0.4$ we know that A still has a *significant* effect on B, but the percentage value of removed uncertainty is only 2.9%.

The lesson from this is that while the values of the measures increase monotonously when uncertainty is reduced, the *scale* of the actual values is *not* linear and *not* intuitive. This needs to be considered when choosing a threshold for when a connection is considered "strong".

6.4 DETECTING DETERMINISTIC RELATIONSHIPS

This section illustrates interesting properties of the Link Strength *Percentages* for deterministic functions. By deterministic function we mean that the state of a child is completely known if the states of all of its parents are known, i.e. there is *no* uncertainty involved.

Definition A node Y is a *deterministic child* of its parents, $P_1, \ldots, P_n$, if

$$\forall \text{ states } y, \forall \text{ parent states } p_1, \ldots, p_n : \quad P(y | p_1, \ldots, p_n) \in \{0, 1\}.$$

Table 2: Connection and Link Strengths for varying b in Figure 2.

b	0.0	0.01	0.02	0.05	0.1	0.2	0.3	0.4	0.5
$MI(A,B) = LS^{true/blind}(A \to B)$	1.0	0.919	0.859	0.714	0.531	0.278	0.119	0.029	0
$MI\%(A,B) = LS\%^{true/blind}(A \to B)$	100	91.9	85.9	71.4	53.1	27.8	11.9	2.9	0.0

Proposition 6.2 *If Y is a deterministic child of its parents, then both its True Average and Blind Average Link Strength Percentage from any parent P is 100%:*

$$\forall P \in parents(Y): \quad LS^{true}\%(P \to Y) = 100\%$$
$$\forall P \in parents(Y): \quad LS^{blind}\%(P \to Y) = 100\%.$$

Proof If node Y is a deterministic child of its parents then it follows $U(Y|X,\mathbf{Z}) = 0$ and $\hat{U}(Y|X,\mathbf{Z}) = 0$ in the definitions of True/Blind Average Link Strengths, which then yields the desired result.

Proposition 6.3 *If $LS^{blind}\%(P \to Y) = 100\%$ for at least one parent P of a node Y, then Y is a deterministic child of its parents.*

Proof From $LS^{blind}\%(P \to Y) = 100\%$ follows $\hat{U}(Y|X,\mathbf{Z}) = 0$, thus

$$\sum_{x,y,\mathbf{z}} P(y|x,\mathbf{z}) \log_2 P(y|x,\mathbf{z}) = 0.$$

Each term $P(y|x,\mathbf{z}) \log_2 P(y|x,\mathbf{z})$ is positive and vanishes if and only if $P(y|x,\mathbf{z}) = 0$ or $P(y|x,\mathbf{z}) = 1$. Thus in order for the whole sum to vanish, we must have $\forall x,y,\mathbf{z}: \; P(y|x,\mathbf{z}) \in \{0,1\}$. Thus Y is a deterministic child of its parents.

Remark: it follows that if $LS^{blind}\%(P \to Y) = 100\%$ for *one* of Y's parents, that the same must hold for *all* of Y's parents.

Proposition 6.4 *Even if $LS^{true}\%(P \to Y) = 100\%$ for all parents P of node Y, then Y is not necessarily a deterministic child of its parents.*

Proof Consider the following counter example. Y has two parents, X, Z, which each can only take states 0 and 1. Let us say that $x = 0$ and $z = 0$ always, thus $P(x = 0, z = 0) = 1$ and $P(x,z) = 0$ otherwise. Define $Y = (x + z) * (\text{random number})$, then $U(Y|x = 0, z = 0) = 0$ and $U(Y|x,z) \neq 0$ otherwise. Thus all products $P(x,z)U(Y|x,z)$ vanish and $U(Y|X,Z) = 0$, although Y is clearly *not* a deterministic child of its parents.

The inability of the True Average Link Strength Percentage to guarantee that a node is a deterministic child comes from the fact that the definition of whether a child is deterministic is *independent of the joint probability of the node's parents*, while True Average Link Strength Percentage *disregards parent state combinations with zero joint probability*. Thus one may argue that this difference is philosophical in nature and that True Average Link Strength Percentage is also a good indicator for deterministic relationships. Nevertheless, it is more prudent to use Blind Average Link Strength Percentage for that purpose.

Let us consider a specific example to see the usefulness in particular of Proposition 6.3. We use the *Visit to Asia* network introduced in (Lauritzen and Spiegelhalter 1998). It represents a simplified version of a medical model and should *not* be used for any medical decisions. Figure 4, 5 and 6 show True Average Link Strength, Blind Average Link Strength and some selected Mutual Information graphs. In the link strength graphs, the value of the link strength is indicated both by the number next to the arrow and by the gray scale of the arrow (if the arrow would otherwise be invisible, a dashed light gray line is used instead). The mutual information graphs indicate the target node by an octagonal shape and the mutual information of all other nodes relative to that one is indicated both by the value underneath each node and by its gray scale.

Looking at the plot for the Blind Average Link Strength Percentage (left plot in Figures 5) immediately shows that *CancerOrTuberculosis* is a deterministic child of its parents – which, admittedly, in this case could have been guessed from its name, too. Other cases are less obvious, in particular if a network is learned from data and this property can be helpful to identify redundant nodes.

6.5 TRUE VERSUS BLIND AVERAGE LINK STRENGTH

Let us revisit the Visit to Asia Example. As indicated by the True Average Percentages on the right of Figure 4 most links are quite strong. Keeping the comments on scale from Section 6.3 in mind all connections except for the one from *Visit to Asia* to *Tuberculosis* can be classified as significant.

The Blind Average Value Percentage for *Visit to Asia* is much higher though, indicating that the reason for the low True Average Percentage is the low probability of state $True$ for *Visit to Asia*. In a nutshell, one could say that in this example **True Average Link Strength (and Percentage) only considers the**

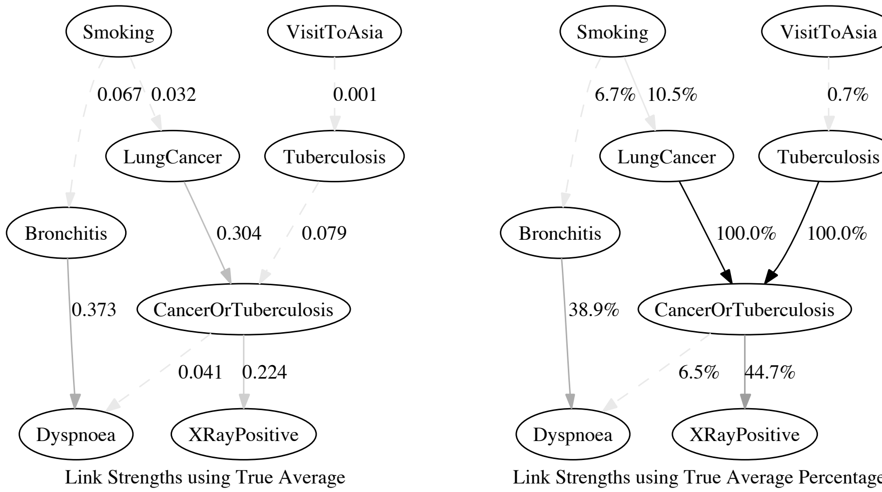


Figure 4: True Average Link Strength (left) and Percentage (right) for Asia Model.

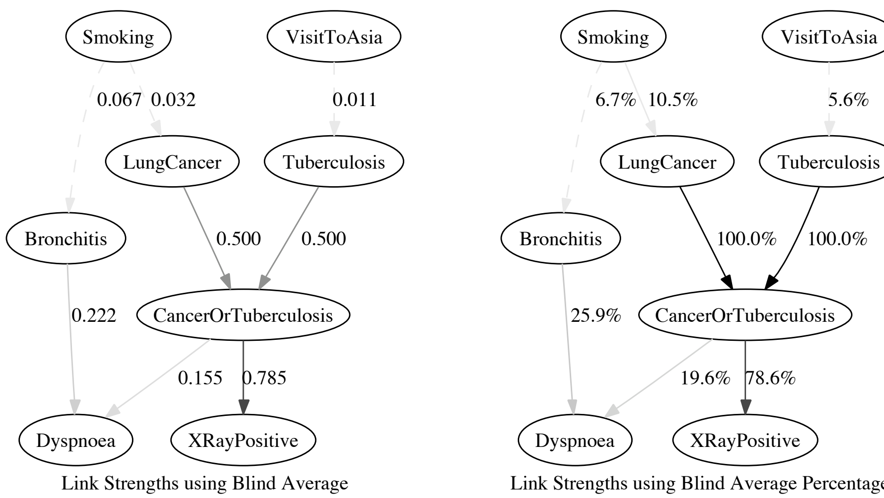


Figure 5: Blind Average Link Strength (left) and Percentage (right) for Asia Model.

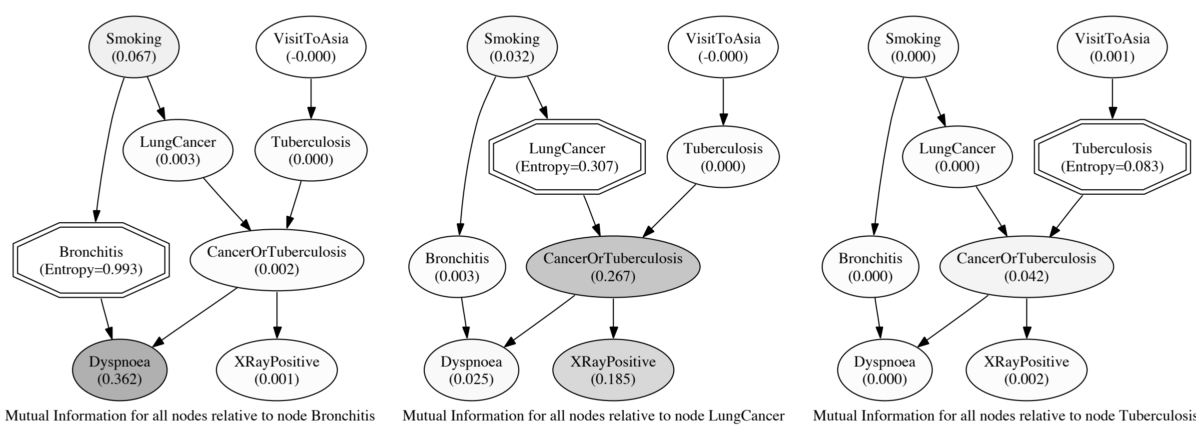


Figure 6: Connection Strength (Mutual Information) relative to node *Bronchitis* (left), *LungCancer* (center) and *Tuberculosis* (right) for Asia Model.

benefit of the information of variable *Visit to Asia* **for the** *average* **patient**. **In contrast Blind Average Link Strength (and Percentage) considers all patient categories equally** – in this case the small group of patients actually having traveled to Asia is given equal weight to the large group not having traveled there – and thus gives more attention to special cases (small groups) and the value of information of variable *Visit to Asia* for that special group.

This difference is typical of the different viewpoints of True Average and Blind Average. One should be aware of those viewpoints when choosing one measure for a particular application.

6.6 COMPARING OUR LINK STRENGTH TO NICHOLSON AND JITNAH'S

Table 3 provides the values for the two link strength measures developed here and for the one from (Nicholson and Jitnah 1998) for two arcs of the Visit to Asia network. For this example they are of similar magnitude. One may have expected that ω would lie between LS^{true} and LS^{blind}, but as the values show, that is not always the case.

Table 3: Comparison of New Link Strength Measures to the One From (Nicholson and Jitnah 1998).

	LS^{true}	LS^{blind}	ω
Visit to Asia → Tub.	0.001	0.011	0.009
Tub. → CancerOrTub.	0.079	0.500	0.602

7 CONCLUSIONS

Much work remains to be done to develop more interpretation and specific guidelines for the use of the measures presented here. Many questions about alternative measures also arise: Are there other measures that have a more intuitive scale? Which other functions $U(X)$ (other than entropy) would be suitable as basis for these measures? In spite of all these questions, we hope that this document will help to foster the discussion on measuring connection and link strengths in a way that is truely meaningful and well understood by the user.

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